Problem 11: Matrices: Sum

Given a square matrix $m[1..n, 1..n]$, compute the sum of the lower half triangle.

$$\begin{array}{rcl} A & = & \underset{m}{M} \times \underset{s}{\mathbb{Z}} \;|\times\; \underset{i}{\mathbb{Z}} \times \underset{j}{\mathbb{Z}} \\ B & = & \underset{m'}{M} \\ Q & = & (m' = m) \\ R & = & Q \wedge s = \sum_{k=1}^{n} \sum_{l=1}^{k} m[k,l] \end{array}$$

Solution

The formal presentation of the problem above more or less contains the solution to this problem. We can compute s with a nested loop:

$$\begin{array}{rcl} P_1 & = & Q \wedge i \in [0,n] \wedge s = \sum_{k=1}^{i} \sum_{l=1}^{k} m[k,l] \\ \pi_1 & = & (i \neq n) \\ t_1 & = & (n - i) \end{array}$$

P_1 is reached from Q via the intermediate state $Q'_1 = (Q \wedge i = 0 \wedge s = 0) \Rightarrow P_1$. Let's solve P_1 for increasing i:

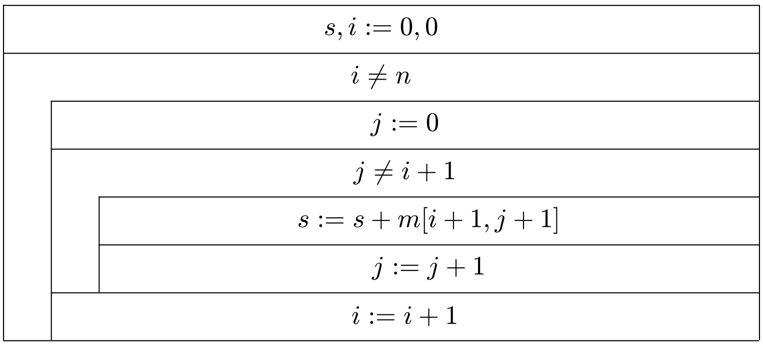
$$\begin{array}{rcl} P_1^{i \leftarrow i+1} & = & Q \wedge (i+1) \in [0,n] \wedge s = \sum_{k=1}^{i} \sum_{l=1}^{k} m[k,l] + \sum_{l=1}^{i+1} m[i+1,l] \\ P_1^{i \leftarrow i+1} & \simeq & P_1 \wedge s = s + \sum_{l=1}^{i+1} m[i+1,l] \end{array}$$

We compute $\sum_{l=1}^{i+1} m[i+1,l]$ with nesting the following loop inside P_1:

$$\begin{array}{rcl} P_2 & = & Q \wedge (i+1) \in [0,n] \wedge j \in [0,i] \wedge s = \sum_{k=1}^{i} \sum_{l=1}^{k} m[k,l] + \sum_{l=1}^{j} m[i+1,l] \\ \pi_2 & = & (j \neq i+1) \\ t_2 & = & ((i+1) - j) \\ Q'_2 & = & Q \wedge (i+1) \in [0,n] \wedge j = 0 \wedge s = \sum_{k=1}^{i} \sum_{l=1}^{k} m[k,l] \end{array}$$

$$\begin{array}{rcl} P_2^{j \leftarrow j+1} & = & Q \wedge (i+1) \in [0,n] \wedge (j+1) \in [0,i] \wedge s = \sum_{k=1}^{i} \sum_{l=1}^{k} m[k,l] + \sum_{l=1}^{j} m[i+1,l] + \\ & & + m[i+1,j+1] \\ P_2^{i \leftarrow i+1} & \simeq & P_2 \wedge s = s + m[i+1,j+1] \end{array}$$

At last, the expression $s = s + m[i+1,j+1]$ can be computed, thus leading to the following program:



$s, i := 0, 0$
$i \neq n$
$j := 0$
$j \neq i + 1$
$s := s + m[i + 1, j + 1]$
$j := j + 1$
$i := i + 1$